AMS 345/CSE 355 (Fall, 2022) Joe Mitchell

COMPUTATIONAL GEOMETRY
Homework Set # 6

Due (upload to Brightspace) on Thursday, November 10, 2022.

Relevant Lecture Modules: 25-28

Recommended Reading: O'Rourke, Chapter 5 (especially 5.2-5.5, and 5.7); Devadoss-O'Rourke, Chapter 4.

In all of the exercises, be sure to give at least a brief explanation or justification for each claim that you make.

PROBLEMS TO TURN IN:

(1). [5 points] (a). Let S be a set of n distinct points in the (Euclidean) plane. The points are not assumed to be in general position (we allow degeneracies).

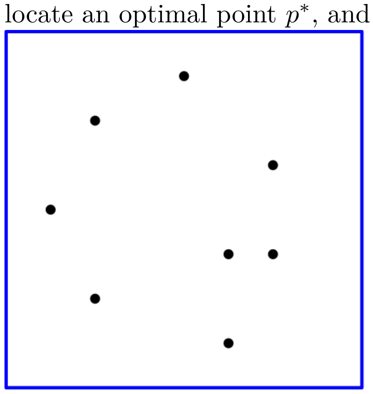
(i). Let $N_{Del}(n)$ be the maximum possible number of edges in a Delaunay diagram of n (distinct) points. Determine the values of $N_{Del}(3)$, $N_{Del}(4)$, $N_{Del}(5)$, and draw examples of sets of $n = 3, 4, 5$ points that achieve these values (sizes of the Delaunay). What is $N_{Del}(n)$?

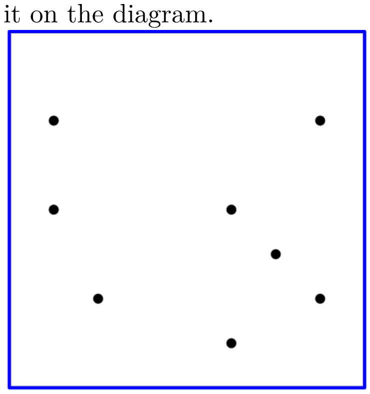
(ii). Let $\nu_{Del}(n)$ be the minimum possible number of edges in a Delaunay diagram of n (distinct) points. Determine the values of $\nu_{Del}(3)$, $\nu_{Del}(4)$, $\nu_{Del}(5)$, and draw examples of sets of $n = 3, 4, 5$ points that achieve these values (sizes of the Delaunay). What is $\nu_{Del}(n)$?

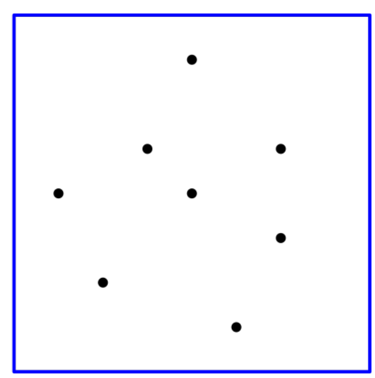
(iii). Let $N_{NNG}(n)$ be the maximum possible number of (directed) edges in a NNG (nearest neighbor graph) of n (distinct) points. Determine the values of $N_{NNG}(3)$, $N_{NNG}(4)$, $N_{NNG}(5)$, and draw examples of sets of $n = 3, 4, 5$ points that achieve these values (sizes of the NNG). (Optional question for extra thought (not graded): What do you think is $N_{NNG}(n)$?)

(iv). Let $\nu_{NNG}(n)$ be the minimum possible number of (directed) edges in a NNG (nearest neighbor graph) of n (distinct) points. Determine the values of $\nu_{NNG}(3)$, $\nu_{NNG}(4)$, $\nu_{NNG}(5)$, and draw examples of sets of $n = 3, 4, 5$ points that achieve these values (sizes of the NNG). What is $\nu_{NNG}(n)$?

(b). Consider a set S of n points (houses) within the unit square (i.e., the square with vertices (0,0), (0,1), (1,1), (1,0)). We want to find a point p^* within the unit square (anywhere in the square) that is as far away from all of the points of S as possible, so that we can place a toxic waste dump at p^*. Describe how you could compute p^* using ideas and methods we have learned. What is the running time (in big-Oh) within which this problem can be solved? For the 3 examples below, in which the unit square is shown in blue, locate an optimal point p^*, and show it on the diagram.

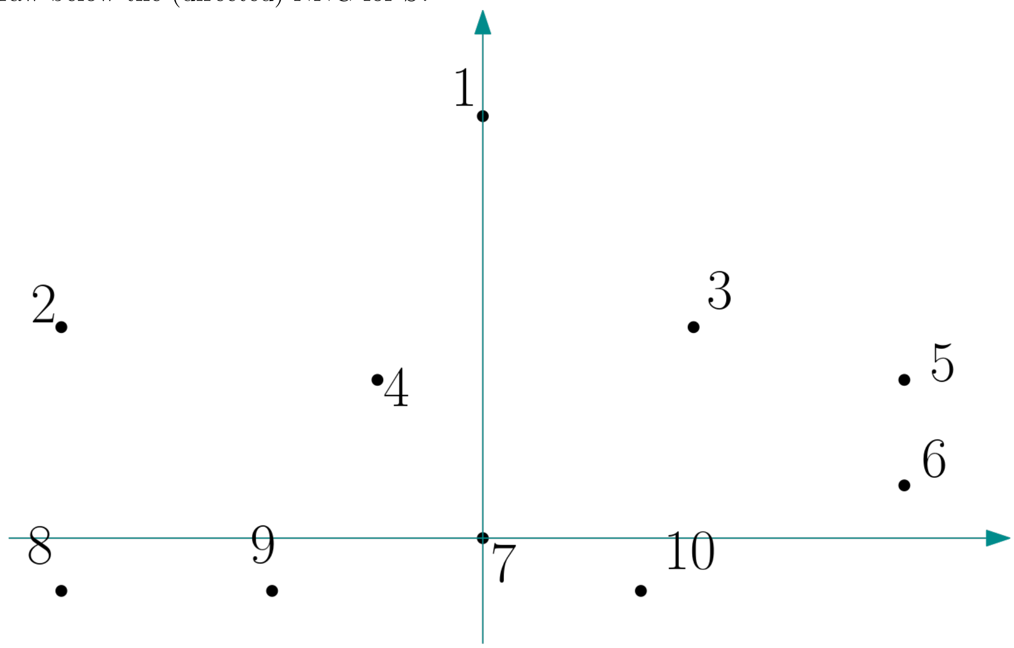




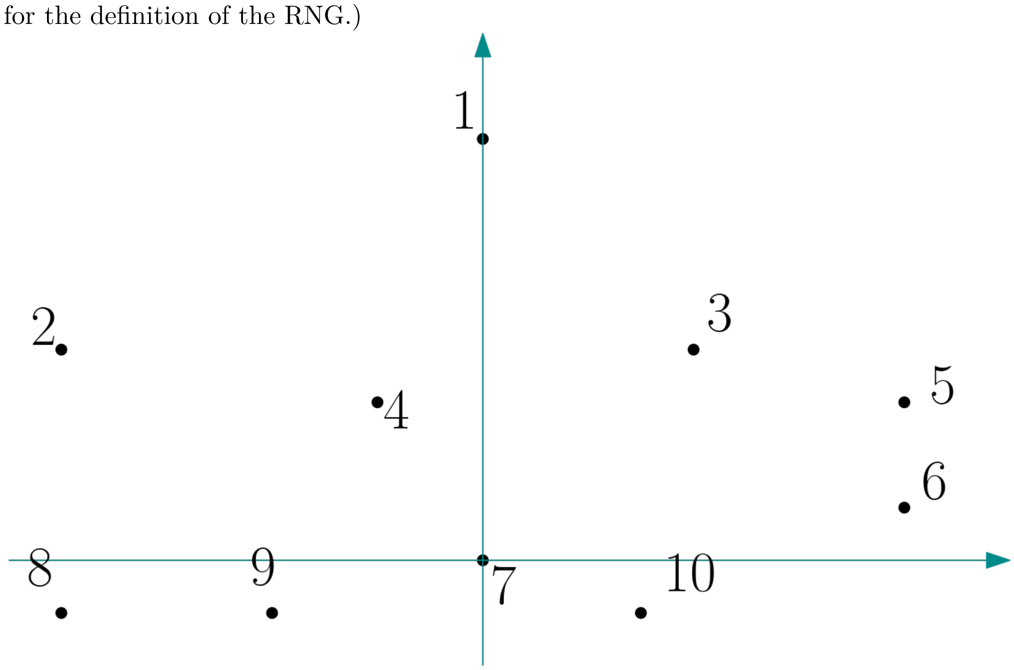


(2). [5 points] (a). Let $S = \{p_1, p_2, \ldots, p_{10}\}$ be the set of points { (0,8), (-8,4), (4,4), (-2,3) (8,3), (8,1), (0,0), (-8,-1), (-4,-1), (3,-1) }, indexed in this order. Below, the points are plotted for you.

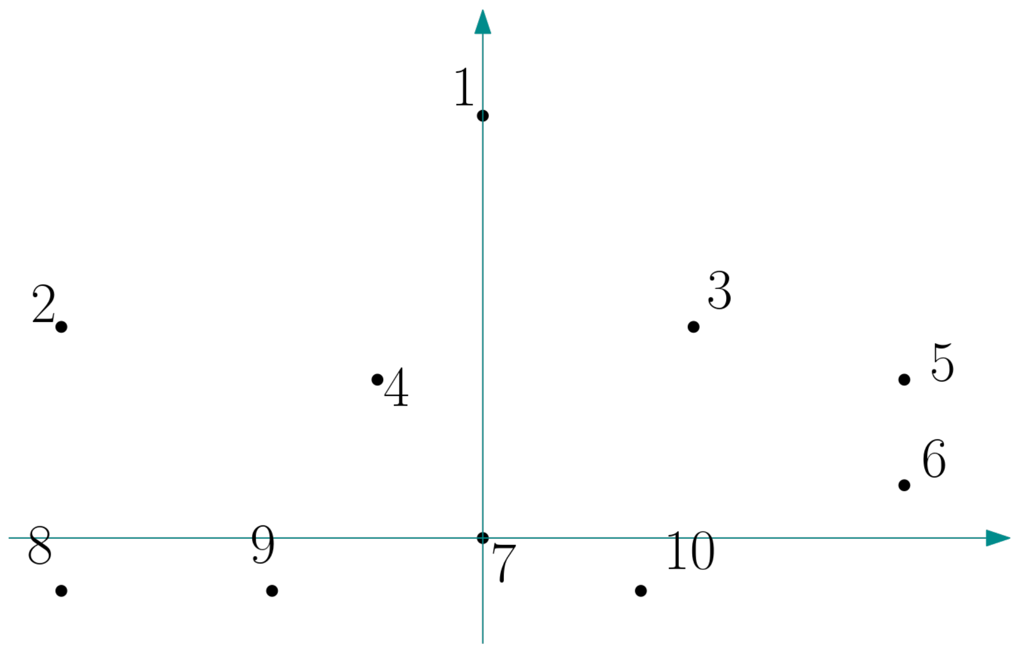
(a). Draw below the (directed) NNG for S.



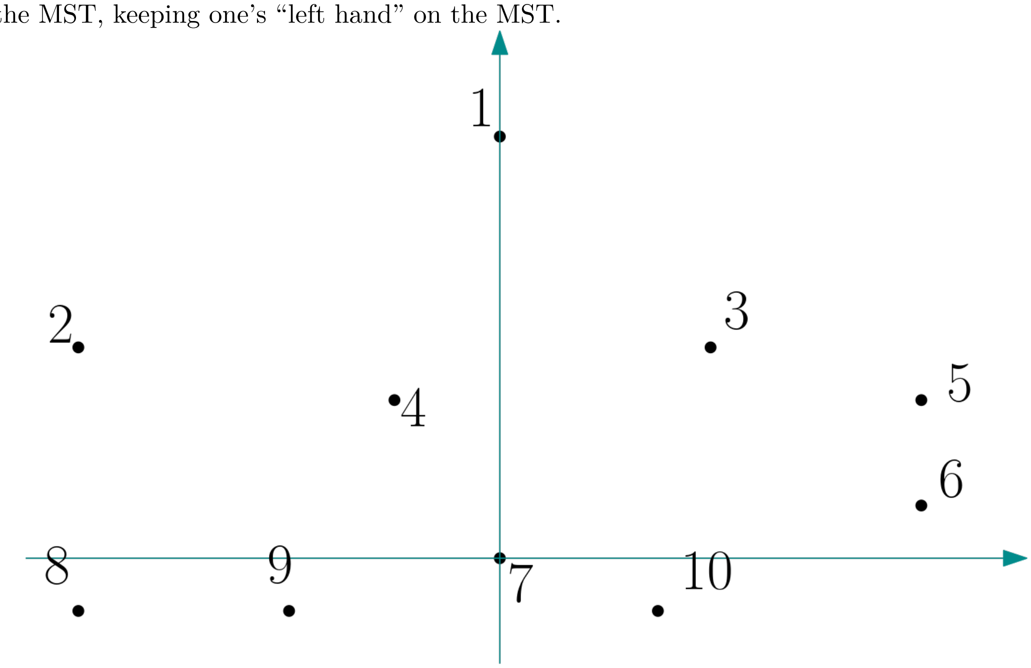
(b). Draw below the relative neighborhood graph (RNG) for S. (See O'Rourke, problem 7, section 5.5.6, page 178, for the definition of the RNG.)



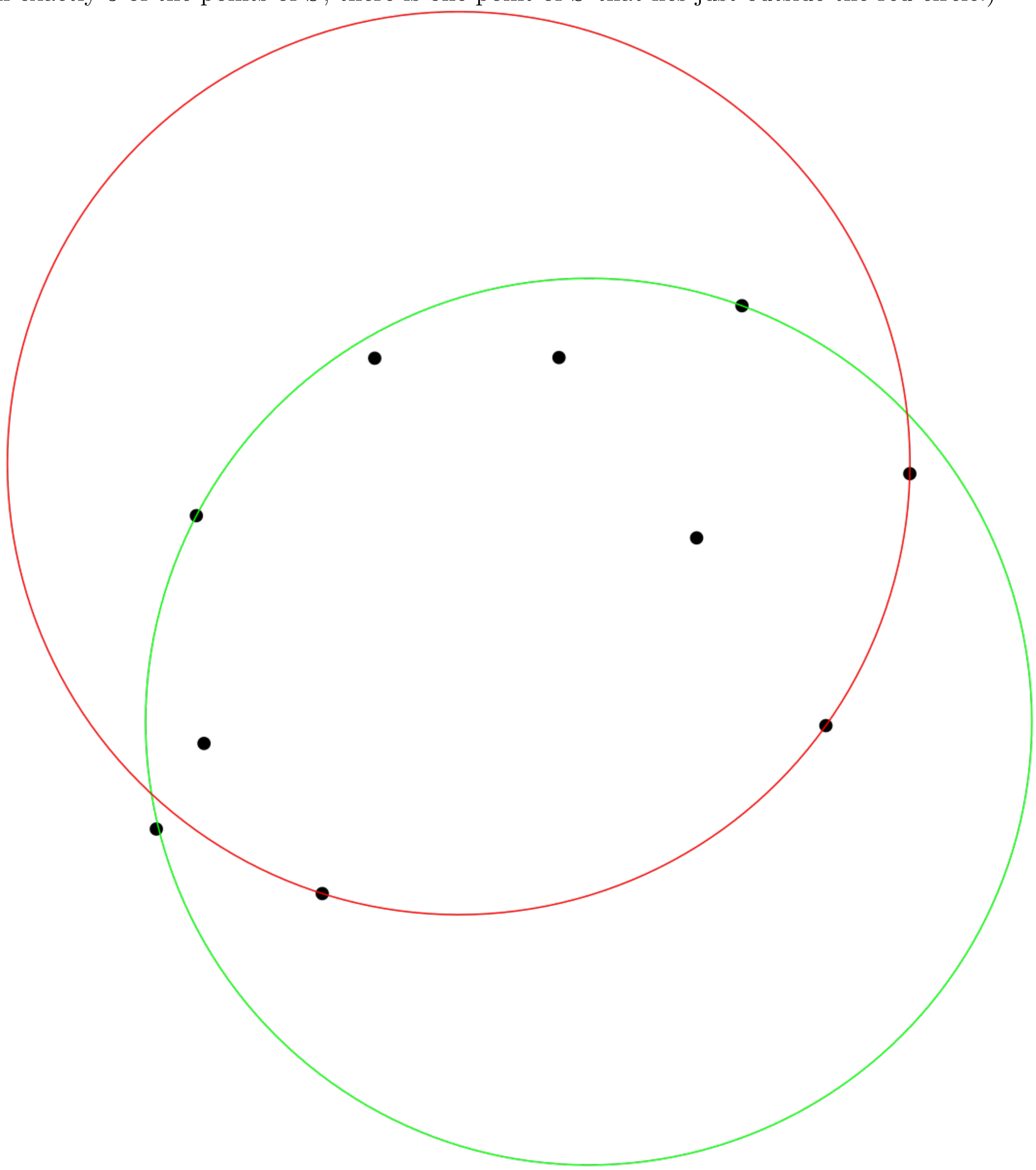
(c). Construct the MST by Kruskal's algorithm applied to the Delaunay diagram. Specifically, in what order are edges added by Kruskal's algorithm? (Give the edges, in order; the edge joining point p_i to point p_j is denoted (i, j).) If there are ties, break them arbitrarily (there may not be a unique MST).



(d). Construct the approximate TSP tour obtained by doubling the MST and shortcutting, as we did in examples in class. (Begin the walk around the MST at point (0,8), labelled “1”, and go counter-clockwise around the MST, keeping one’s “left hand” on the MST.



(e). Construct (in blue) the "furthest-neighbor Voronoi diagram" (FNVD, also known as the "furthest-point Voronoi diagram") of the set S of points below. Start by constructing (in red) the "furthest-neighbor Delaunay diagram" (FNDD, also known as the "furthest-point Delaunay diagram") of S. Two circles are shown to help you reason about the points and the FNDD. (Each of the two circles (green and red) pass through exactly 3 of the points of S; there is one point of S that lies just outside the red circle.)



ADDITIONAL PRACTICE PROBLEMS: TRY THEM, UNDERSTAND THEM, YOU ARE RESPONSIBLE FOR THEM

(3). Let S be a set of n points in the plane in general position (no 3 are collinear, no 4 are cocircular). Let i denote the number of points of S that are strictly interior to the convex hull, $CH(S)$.

(a). Consider the Delaunay triangulation, $\mathcal{D}(S)$. As a function of n and i how many triangles does $\mathcal{D}(S)$ have? How many Delaunay edges are there in $\mathcal{D}(S)$? How many Voronoi vertices does the Voronoi diagram of S have?

(b). (no relationship to part (a)) Now we are interested in decomposing the convex hull of S into "quadrangles" (4-sided polygons (*quadrilaterals*), not necessarily convex), such that each point of S is a vertex of some quadrangle.

(i). It turns out that it is not always possible to "quadrangulate" the convex hull. Let h be the number of vertices of the convex hull, $CH(S)$. Show that for any set S of n sites, if h is odd, then there is no quadrangulation of the convex hull of S.

(ii). It turns out that if h is even, there is always a quadrangulation of S. (Extra Challenge: Think about how you might prove this (no proof is required for the assignment, though).) Suppose now that S is such that h is even. As a function of n and h, how many quadrangles must be in any quadrangulation of the convex hull of S? How many edges are there in such a quadrangulation? (Hint: Try small values of h, n to check yourself.

(Note: It is open to find an algorithm that will discover if there is a *convex* quadrangulation of S (in which each quadrangle is convex).)

(4). Recall that the NNG is a directed graph that connects a point p_i to a point p_j, with a directed edge (p_i, p_j), if and only if p_j is a nearest neighbor of p_i. (Note that p_i may have multiple nearest neighbors, tied in distance from p_i.) The out-degree of p_i in the NNG is the number of different directed edges, (p_i, p_j), leading out of p_i in NNG; the in-degree of p_i is the number of different directed edges, (p_j, p_i), leading into p_i.

(a). Give an example of a set of n points in the plane such that in the NNG there is one (of the n) points that has out-degree $n-1$.

(Question for extra thought [optional]: Is it possible to have two points, each with in-degree $n-1$?)

(b). Draw an example of a point set $S = \{p_1, p_2, \ldots, p_n\}$ such that p_1 has a "high" in-degree. How high can you make it? (Try to make it as large as possible in your example.)

(5). The Delaunay triangulation of a set S of points in the plane is "optimal" in some respects. In particular, we know that it maximizes the minimum angle among the angles of the triangles. (Assume here that the Delaunay diagram of S is a triangulation; e.g., you may assume that no four points are cocircular.)

(a). Does the Delaunay triangulation *minimize the maximum angle*? In other words, if the "score" of a triangulation of S is the maximum interior angle among all of its triangles, does the Delaunay triangulation have the lowest "score" among all possible triangulations of S?

(b). Does the Delaunay triangulation minimize the total length of the edges of the triangulation?

In both cases, justify your answer! Either give an argument why it is true, or give a counterexample to show that it can be false (making sure to draw appropriate circles to show that the edges you claim are Delaunay really are).

(6). Let S be the set of points {(-3,-1), (0,3), (3,7), (6,7), (7,3), (10,3), (10,-1), (6,-1), (3,-1)}. See the figure below.

(a). Draw the (directed) NNG for S.

(b). Draw the relative neighborhood graph (RNG) for S. (See O'Rourke, problem 7, section 5.5.6, page 178, for the definition of the RNG.)

(c). Construct the MST by Kruskal's algorithm applied to the Delaunay diagram. Specifically, in what order are edges added by Kruskal's algorithm? (Give the edges, in order; the edge joining point i to point j is denoted (i, j).) If there are ties, break them arbitrarily (there may not be a unique MST).

(d). Construct the approximate TSP tour obtained by doubling the MST and shortcutting, as in Figure 5.18 of [O'Rourke]. (Begin the walk around the MST at point (-3,-1) and go clockwise around the MST, as done in the example in the text.) Also determine the *optimal* TSP tour (easy to do in this case – can you justify your answer?).

(e). Construct the "furthest-point Voronoi diagram" of S. Read problem 11, section 5.5.6, and see Figure 5.19 for the definition of the diagram.